Simulation of partial entanglement with non-signaling resources

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With the goal of gaining a deeper understanding of quantum non-locality, we decompose quantum correlations into more elementary non-local correlations. We show that the correlations of all pure entangled states of two qubits can be simulated without communication, hence using only non-signaling resources. Our simulation model works in two steps. First, we decompose the quantum correlations into a local and a non-local part. Second, we present a model for simulating the nonlocal part using only non-signaling resources. In our model partially entangled states require more nonlocal resources than maximally entangled states, but the less the state is entangled, the less frequently must the nonlocal resources be used.

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I. INTRODUCTION

Quantum correlations are very peculiar, especially those violating some Bell inequality [1]. Gaining a deeper insight into such nonlocal quantum correlation is a grand challenge. Children gain understanding of how their toys function by dismantling them into pieces. In the present paper we follow a similar approach by decomposing the quantum correlations into simpler, more elementary, nonlocal correlations.

This work is part of the general research program that looks for nonlocal models compatible and incompatible with quantum predictions. The goal is to find out what is essential in quantum correlations. Note that we do not claim that Nature functions as our model. Nevertheless we believe that finding the minimal resources sufficient to simulate quantum correlations, and studying the computational power that they offer [2, 3, 4, 5], provide enlightening insights into the quantum world.

In the last years, two different ways of decomposing quantum correlations have been proposed. The first one, due to Elitzur, Popescu and Rohrlich (whence EPR-2 !) [6], consists in decomposing some quantum correlations into a local and a non-local part. A second approach consists in the simulation of entanglement with the help of some non-local resource, e.g. classical communication or a non-local box. While communication models [7, 8] give insight to quantum correlations from the point of view of communication complexity, we believe that models using only no-signaling resources [9] are more relevant from a physical point of view, since it is most unlikely that Nature uses any form of communication [10]. In this paper, we shall combine for the first time both approaches, and prove that all pure entangled states of two qubits can be simulated using only no-signaling resources, i.e. without communication.

The approach we follow works in two steps. First, in the EPR-2 spirit, we decompose the quantum correlation P_Q corresponding to von Neumann measurements performed on pure entangled states of two qubits $|\psi(\theta)\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$ into a statistical mixture of a local correlation P_L and a nonlocal correlation P_{NL} [6]:

$$P_Q = p_L(\theta)P_L + (1 - p_L(\theta))P_{NL} \ . \quad (1)$$

The weight $p_L(\theta)$ is thus a measure of the locality of the state $|\psi(\theta)\rangle$. In particular, for any maximally entangled state of two-qubits one has $p_L(\theta = \pi/4) = 0$ [6], a result that holds true for maximally entangled states in any dimension [11]. Note that in general the probability distribution P_{NL} does not need to be quantum, but is restricted to no-signaling correlations by construction.

Then, we provide a simulation of the nonlocal correlation P_{NL} using only nonlocal, but non-signaling resources. Accordingly, in order to simulate P_Q, it suffice to simulate P_L with probability $p_L(\theta)$, which requires only shared randomness (but no nonlocal resources), and to simulate P_{NL} with the complementary probability $1 - p_L(\theta)$. As expected, the less the quantum state $|\psi(\theta)\rangle$ is entangled, the smaller the weight $p_L(\theta)$ of the local correlation [6, 12]. Consequently, the simulation of a less entangled state requires less frequent use of nonlocal resources; in particular for separable states $p_L(\theta = 0) = 1$. However, not much is known about the nonlocal resources needed to simulate the nonlocal part of quantum correlations, i.e. to simulate p_{NL}. Reference [9] presented a simulation of the quantum correlation for the special case of maximally entangled qubit pairs ($\theta = \pi/4$) using only one nonlocal box, the so-called PR-box [13]. For non-maximally entangled qubit states, very few is known. To our knowledge, the only known result is that one PR-box is not sufficient for simulating slightly entangled states [14]. This result shows that entanglement and non-locality are different resources, as also suggested by other works [15, 16, 17, 18].

In this paper we use a decomposition of the form (1), recently presented in Ref. [12], which is optimal under some general assumption, and present a simulation of the corresponding nonlocal correlation P_{NL} for arbitrarily entangled two qubit states. This simulation requires finitely many nonlocal boxes, though no claim of optimality can be made. For pedagogical reasons, the paper is organized as follows. After introducing the general framework in Section 2, and briefly reviewing the case of maximal entanglement in Section 3, we

present in Section 4 a preliminary model for simulating partially entangled qubit states, without using any decomposition into local and nonlocal parts. This allows us to introduce the two main ingredients of our model: first the technique of correlated local flips; second the Millionaire box, a generalization of the PR-box. Then in Section 5, we briefly recall the decomposition into local and non-local parts presented in [12], and explain how our preliminary simulation model can be extended to simulate the nonlocal part P_{NL} of the model of Ref. [12]. Finally we give some conclusions and perspectives.

II. GENERAL FRAMEWORK

Formally, a correlation is a conditional probability distribution $P(\alpha\beta|\vec{a}\vec{b})$, where α, β denote the outcomes observed by Alice and Bob when they perform measurements labeled by $\vec{a}$ and $\vec{b}$. Here, measurements are conveniently represented as vectors on the Bloch sphere, since we focus on von Neumann measurements on qubits. A correlation is nonsignalling if and only if Alice and Bob's marginals M_A and M_B, are independent of the partner's input: M_A does not depend on $\vec{b}$ and M_B does not depend on $\vec{a}$. For binary outcomes ($\alpha, \beta \in \{-1,+1\}$), the correlations are conveniently written as

$$P(\alpha,\beta|\vec{a},\vec{b}) = \frac{1}{4}\left(1+\alpha M_A(\vec{a})+\beta M_B(\vec{b})+\alpha\beta C(\vec{a},\vec{b})\right) \quad (2)$$

where

$$\begin{aligned} M_A(\vec{a}) &= \sum_{\alpha,\beta}\alpha P(\alpha,\beta|\vec{a},\vec{b}) \\ M_B(\vec{b}) &= \sum_{\alpha,\beta}\beta P(\alpha,\beta|\vec{a},\vec{b}) \ , \end{aligned} \quad (3)$$

are the local marginals, and

$$C(\vec{a},\vec{b}) = \sum_{\alpha,\beta}\alpha\beta P(\alpha,\beta|\vec{a},\vec{b}) \quad (4)$$

is the correlation term. Here we shall focus on pure entangled states of two qubits $|\psi(\theta)\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$, $\theta \in]0,\pi/4]$. Thus the quantum correlation $P_Q(\alpha\beta|\vec{a}\vec{b})$ is given by

$$\begin{gathered} M_A(\vec{a}) = ca_z \ , \ M_B(\vec{b}) = cb_z \\ C(\vec{a},\vec{b}) = a_zb_z + s(a_xb_x - a_yb_y)\,, \end{gathered} \quad (5)$$

where $c \equiv \cos 2\theta$ and $s \equiv \sin 2\theta$.

Now, we would like to decompose the correlations P_Q into simpler ones, such that

$$P_Q(\alpha,\beta|\vec{a},\vec{b}) = \int d\lambda P_\lambda(\alpha,\beta|\vec{a},\vec{b}) \quad , \quad (6)$$

where $d\lambda$ is a normalized measure. In a simulation model, two ingredients are required: first, a non-local resources for creating the elementary correlations P_λ; second, a strategy (represented by the λ's) for judiciously combining them. In this paper, we provide such a decomposition. The remarkable feature of our model is that the elementary nonlocal correlations are obtained without communication, that is using only no-signaling resources.

III. MAXIMALLY ENTANGLED STATE

Let us briefly the simple case of maximal entanglement, i.e. $\theta = \pi/4$. In this case the marginals vanish, $M_A(\vec{a}) = M_B(\vec{b}) = 0$, and the correlation takes the simple scalar product form $C(\vec{a},\vec{b}) = \vec{a}\cdot\vec{b}$ [32]. In reference [9] a model simulating this correlations is presented. This model uses as resources only shared randomness and one PR-box, which satisfies the relation $a \oplus b = xy$, where x, y are Alice's and Bob's input bits, and a, b their outcome bits. In general nonlocal boxes provide some elementary nonlocal correlations. They are elementary in that they allow only for a limited (usually finite) number of inputs and outputs and they are extremal points in the convex set of nonsignaling correlations [19]. They are nonlocal in the sense that they violate some Bell inequality. Importantly, they do not allow signalling, that is the statistics of the local outcomes (i.e. the marginals) are independent from the other parties inputs. This model demonstrates that the resource needed to simulate maximally entangled qubit pairs is surprisingly simple. Indeed, what could be simpler than $a \oplus b = xy$?

IV. PARTIALLY ENTANGLED STATES: PRELIMINARY MODEL

We now turn to partially entangled states of two qubits. In general the marginals $M_A(\vec{a})$ and $M_B(\vec{b})$ do not vanish. However, in the case of two parties and binary outcomes, it is proven that all extremal nonlocal boxes have vanishing (or deterministic) marginals [20, 21]. This explains in part why it is difficult to simulate partially entangled states. In order to circumvent this difficulty we introduce now the concept of correlated local flips.

A. Correlated local flips.

Let us consider an arbitrary probability distribution

$$P_0(\alpha,\beta|\vec{a},\vec{b}) = \frac{1}{4}(1+\alpha\beta C_0(\vec{a},\vec{b})) \ , \quad (7)$$

with vanishing marginals and correlation term $C_0(\vec{a},\vec{b})$. Now, Alice and Bob perform local flips on the probability distribution P_0; that is, Alice (Bob) flips her (his) output -1 with a probability f_a (f_b), while the output $+1$ is left untouched. After this processing, also called a Z-channel, the marginals are

clearly biased towards +1. Let us now assume that $f_b \geq f_a$ and that the flips of Alice and Bob are both determined by a shared random variable Λ uniformly distributed in $[0,1]$. Alice and Bob flip their -1 outcome if and only if $\Lambda < f_a$ and $\Lambda < f_b$, respectively. The resulting probability distribution reads

$$P_f(\alpha,\beta|\vec{a},\vec{b}) = \frac{1}{4}(1+\alpha f_a+\beta f_b+\alpha\beta(f_a+(1-f_b)C_0(\vec{a},\vec{b})))\,. \tag{8}$$

It should be pointed out that the flips f_a and f_b must be correlated; this will be crucial in the following. Note also that every probability distribution $P(\alpha,\beta)=\frac{1}{4}(1+\alpha M_A+\beta M_B+\alpha\beta C)$ with $M_B \geq M_A$ can be generated in this way.

B. Preliminary model, step 1

We just described a technique for creating a probability distribution P_f with nontrivial (i.e. non vanishing) marginals, starting from an initial probability distribution P_0 which had trivial marginals. Now the intuition is the following: since correlation with trivial marginals seem to be easier to create with standard nonlocal resource (such as PR-boxes), let us do the identification $P_f = P_Q$ and find out what is the required initial distribution P_0. For partially entangled states of two-qubits (P_Q given by (5)), this leads to

$$f_a = ca_z\ ,\ f_b = cb_z\ ,\ C_0 = \vec{a}\cdot\vec{B} \tag{9}$$

where

$$\vec{B} \equiv (sb_x, -sb_y, b_z - c)/(1-cb_z)\ . \tag{10}$$

Note that $||\vec{B}|| = 1$. Remarkably, $\vec{B}$ corresponds to Bob's original measurement setting $\vec{b}$ moved one step back on the Hardy ladder [22].

Consequently the problem of simulating correlations originating from von Neumann measurements on partially entangled states reduces to the problem of simulating the unbiased probability distribution

$$P_0 = \frac{1}{4}(1+\alpha\beta\vec{a}\cdot\vec{B})\ . \tag{11}$$

Such a "scalar product" correlation can be reproduced with a single bit of communication [7] or with a single PR-box [9]. However, there is a caveat: Alice and Bob must know wether $b_z \geq a_z$ (as assumed above) or if on the contrary $a_z \geq b_z$! This is due to the fact that the local flips must be correlated. Note that in the case $a_z \geq b_z$, the initial probability distribution is given by $P_0 = \frac{1}{4}(1+\alpha\beta\vec{A}\cdot\vec{b})$, where $\vec{A}$ is defined similarly to equation (10).

At first sight it may seem that a resource solving this problem will lead to signaling, because it would reveal a relationship between Alice's and Bob's measurements. Remarkably, this is not the case. Next, we show that a no-signaling (nonlocal) resource known as the Millionaire box is exactly the tool we need.

C. The Millionaire box.

Two millionaires challenge each other: who is richer ? Since millionaires are in general quite reluctant to reveal how much money they own, they prefer to use the Millionaire-box (M-box) [23], a nonlocal two-input two-output non-local box. The two outputs a,b are binary, ($a,b \in \{0,1\}$), and are locally random in order to ensure no-signaling. The two inputs x,y can be chosen in the continuous interval $[0,1]$. The M-box is characterized by the following relation:

$$a \oplus b = [x \leq y]\ , \tag{12}$$

where $[X]$ denotes the logical value of X: $[X]=0$ when X is true. Note that the M-box admits an infinite number of possible inputs. So, both millionaires input the amount of money they own x,y into the machine; the parity of the outputs ($a\oplus b$) indicates the winner. Fortunately, the M-box is also useful to physicists, as will be shown in the next section. Note that the M-box is a generalization of the PR-box; in case the inputs x, y are binary, the M-box is simply equivalent to a PR-box (given here by $x(y+1) = a\oplus b\oplus 1$). It is also worth mentioning that the M-box reaches the no-signaling bound of all the Bell inequalities I_{NN22} introduced in [24]. An interesting question is whether all (bipartite) non-local boxes with two-outcomes [20, 21] can be simulated with one M-box. Indeed, a detailed study of the non-local properties of the M-box would be relevant, but is beyond the scope of this paper.

D. Preliminary model, step 2

As shown above, the technique of local flips allows one to recover the correlation of partially entangled states, under the condition that $b_z \geq a_z$ (or $a_z \geq b_z$). But how do Alice and Bob know whether $b_z \geq a_z$ or $a_z \geq b_z$? The M-box can overcome this problem.

Alice and Bob share two PR-boxes for creating "scalar product" correlations (see Fig. 1); from now on we call these CGMP-boxes [9]. The first one is used to create the correlation given by the scalar product $\vec{a}\cdot\vec{B}$, i.e. corresponding to the case $b_z \geq a_z$ and the second one for the scalar product $\vec{A}\cdot\vec{b}$, i.e. for the case $a_z \geq b_z$. Local flips are then performed. At this point, Alice and Bob have each got two possible outputs α_1,α_2 and β_1,β_2, but don't know which one to use, since they don't know whether $a_z \leq b_z$ or $b_z \leq a_z$.

Next, they input the z-component of their measurement setting (respectively a_z and b_z [33]) into the M-box, and get outputs a and b. It is clear that, for the simulation to succeed, the final output of Alice and Bob, α and β, should be equal to α_1,β_1 if $a_z \geq b_z$, and equal to α_2,β_2 if $b_z \geq a_z$. Mathematically this translates into the following expression

$$\alpha\oplus\beta = (a\oplus b)(\alpha_1\oplus\beta_1)\oplus(a\oplus b\oplus 1)(\alpha_2\oplus\beta_2)\,. \tag{13}$$

Developing the previous equation, one gets

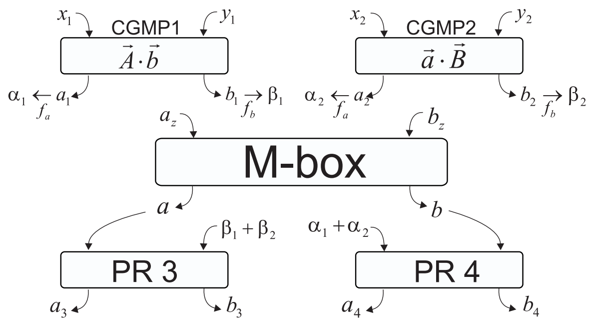

FIG. 1: Preliminary model. Simulating partial entanglement without communication. The model requires four PR-boxes and a Millionaire-box (M-box). The first two PR-boxes create "scalar product" correlations (CGMP-boxes). Then the M-box "selects" the correct CGMP-box, without revealing any relation between Alice's and Bob's measurement settings (i.e. without signaling). Finally, two additional PR-boxes are required for computing the correct outputs.

$$\begin{aligned} \alpha \oplus \beta &= a(\alpha_1 \oplus \alpha_2) \oplus \alpha_2 \oplus b(\beta_1 \oplus \beta_2) \oplus \beta_2 \\ &\quad \oplus a(\beta_1 \oplus \beta_2) \oplus b(\alpha_1 \oplus \alpha_2)\,, \end{aligned} \tag{14}$$

which contains some local terms, as well as some non-local terms. Remarkably, the non-local terms (second line of equation (14)) are simply obtained by using two supplementary PR-boxes, $a_3 \oplus b_3 = a(\beta_1 \oplus \beta_2)$, and $a_4 \oplus b_4 = b(\alpha_1 \oplus \alpha_2)$ (see Fig. 1).

So finally, using four PR-boxes (two CGMP-boxes and two additional PR-boxes) and one M-box, one can simulate the correlation of any partially entangled state of two qubits. Whether the M-box can be replaced by a finite number of PR-boxes (or more generally with a nonlocal box having a finite number of possible inputs) is an interesting open question.

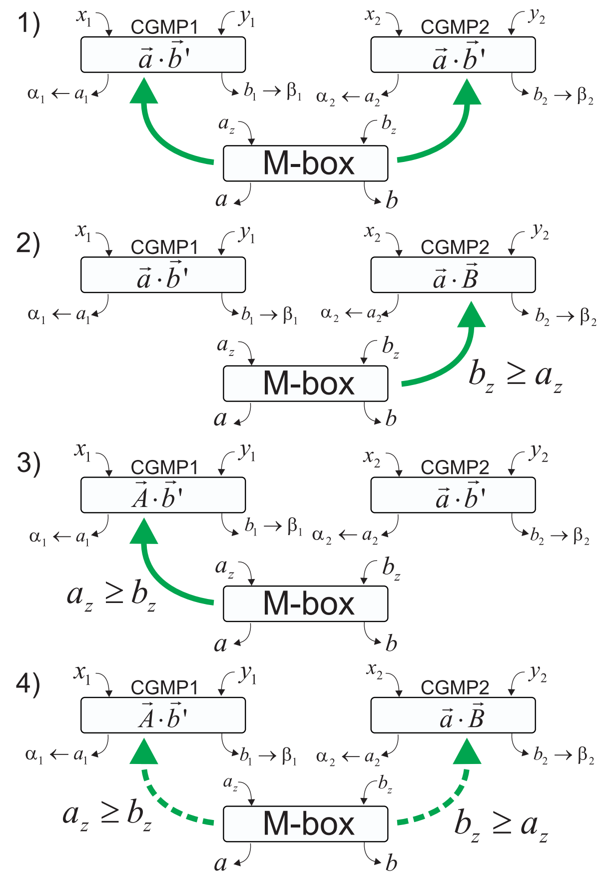

FIG. 2: Sketch of the main model. 1) Alice and Bob inside the slice (see text); both CGMP-boxes can be used. 2) Alice inside, Bob outside; then indeed $a_z \leq b_z$. 3) Alice outside, Bob inside. 4) Alice and Bob outside. Depending wether $a_z \leq b_z$ or $a_z \geq b_z$, the M-box selects the correct CGMP-box. Note that we have omitted the two additional PR-boxes (PR_3, PR_4 of Fig. 2).

V. PARTIALLY ENTANGLED STATES: MAIN MODEL, INTEGRATING EPR-2

We are now ready to present our model, combining the preliminary model (presented in the previous section) and the decomposition of Ref [12], into local and non-local parts (i.e. of the form (1)). The decomposition is the following:

$$\begin{aligned} p_L(\theta) &= 1-s \\ P_L &= \frac{1}{4}\left(1+\alpha f(a_z)\right)\left(1+\beta f(b_z)\right) \\ P_{NL} &= \frac{1}{4}\left(1+\alpha F(a_z)+\beta F(b_z)+\alpha\beta G(\vec{a}\vec{b})\right) \end{aligned} \tag{15}$$

where $f(x) = \mathrm{sgn}(x)\min(1, \frac{c}{1-s}|x|)$, $F(x) = \frac{1}{s}(cx - (1-s)f(x))$, and $G(\vec{a}\vec{b}) = a_x b_x - a_y b_y + \frac{1}{s}[a_z b_z - (1-s)f(a_z)f(b_z)]$. We refer the reader to [12] for further details.

Let us point out two important features of decomposition (15) First, the weight of the local part $p_L(\theta) = 1-s$ is a monotonic decreasing function of θ, i.e. of the degree of entanglement of the state $|\psi(\theta)\rangle$. Note also that $p_L(\theta) = 1-s$ is optimal under the assumption that P_L depends only on a_z and b_z. Second, the non-local part P_{NL} depends on the measurement settings. More precisely, when the measurement setting of Alice is such that $a_z \leq (1-s)/c$ (i.e. inside a slice of the Bloch sphere around the equator), her local marginal vanishes; and similarly for Bob. On the contrary, when the measurement setting lies outside the slice, the marginal is biased. When both the settings of Alice and Bob are found inside the slice, the correlation reduces to a simple scalar product with trivial marginals.

The simulation of P_{NL} is very similar to that presented above, thus we only describe Alice's and Bob's strategies. As previously, the required non-local resources are two CGMP-boxes, an M-box and two additional PR-boxes. After establishing non-local correlations with both CGMP-boxes, Alice and Bob perform local flips. Finally they use two additional PR-boxes to compute the correct output (see Fig. 2).

Alice proceeds as follows. When her setting is inside the

slice ($a_z \leq (1-s)/c$), she inputs according to $\vec{a}$ into both CGMP-boxes, and does not perform any local flip ($f_a = 0$). When her setting is outside the slice, she inputs the first CGMP-box according to $\vec{A} = (sa_x, sa_y, c - a_z)/(1 - a_z c)$ and the second CGMP-box according to $\vec{a}$. Then she biases her output towards outcome +1 with probability $f_a = F(a_z)$.

Bob proceeds almost similarly. When his measurement setting is inside the slice ($b_z \leq (1-s)/c$), he inputs both CGMP-boxes according to $\vec{b}' = (b_x, -b_y, -b_z)$. When his setting is outside the slice, he inputs the first CGMP-box according to $\vec{b}'$ and the second according to $\vec{B} = (sb_x, -sb_y, b_z - c)/(1 - b_z c)$. Then he biases his output with probability $f_b = F(b_z)$.

VI. CONCLUSION AND OUTLOOK

By dismantling the quantum correlations of partially entangled states of two-qubits into more elementary nonlocal but no-signaling correlations, we gained insight into the quantum world. We showed that the correlations of all pure entangled states of two qubits (under von Neumann measurements) can be simulated using only non-signaling resources, hence without communication. Our decomposition is likely not to be optimal in the sense that there might exist more economical models. Still, there are already two lessons we learn from the present decomposition. First, the less the quantum state is entangled, the less frequently one needs to use nonlocal resources to simulate it; as intuition suggests. Next, whenever one needs nonlocal resources, then these are definitively larger for (at least some) partially entangled states than for the maximally entangled state; indeed this is proven for slightly entangled states [14], but is still an open question for close to maximally entangled states. Hence, in counting the resources required to simulate two-qubit states, one should distinguish between the required amount of nonlocal resources and the frequency at which one has to use them.

It is interesting to establish the following connection with Leggett's approach to quantum correlation [25], which recently attracted quite some attention [26, 27, 28, 29, 30, 31]. In models *à la Leggett* one assumes that the elementary correlations, contrary to PR-boxes, have nontrivial marginals; Leggett's original idea is that each qubit, when analysed individually, appears to be always in a pure state, see [25, 29]. However, one can prove that any such model, with elementary correlation having nontrivial marginals, fails to reproduce the quantum correlation of maximally entangled states of two-qubits [29, 31]. This is a kind of converse to the present paper in which we show that it is especially hard to simulated at the same time nonlocal correlations and non-vanishing marginals.

Among the open questions, we like to underline the following one. How could one prove that a decomposition is minimal? As said, this question has two sides. Minimality of the resources, and minimality of the frequency at which one has to use them. Our experience suggests that the first aspect is an especially difficult problem. The second aspect looks more promising: it seems natural to conjecture that an EPR2-type decomposition with $p_{NL}(\theta) = 1 - \cos 2\theta$ should exist [12].

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